PMM U.S.S.R.,Vol.47,No.1,pp.76-81,1984
Printed in Great Britain

0021-8928/84 $10.00+0.00

UDC 539.3

AN APPROACH TO THE SOLUTION OF GEOMETRICALLY NONLINEAR PROBLEMS OF APPLIED SHELL THEORY*

V.G. TROSHIN

A method is proposed for linearization of geometrically nonlinear equations of applied shell theory /1/. The initial nonlinear problem is reduced to a sequence of linear problems for a shell having the additional parameters of flexural rigidity, curvature and torsion; these parameters are determined in the course of successive approximations. To construct the deformation curve, the entire process of applying the load is divided into a series of stages, at each of which the solution is found in a hyperplane perpendicular to a line passing through a points corresponding to the two preceding stages. By means of the algorithm, it is possible to construct the deformation curves for both single and multi-parameter loads.

Depending upon the level at which linearization is performed, existing methods of solving these nonlinear problems may be divided into three basic groups: (1) linearization of a system of algebraic (cf. /2/) and (2) ordinary differential equations /3/ to which the initial two-dimensional problem has been reduced; (3) linearization of a system of partial differential resolvent equations. In a number of cases, there is a third, more preferable approach in which the process of linearization of the initial equations does not involve a particular method of solving a boundary-value problem. There are two basic methods known that implement this approach in general form: simple iteration and successive loadings /4/. In the first case, the nonlinear terms are carried over to the right side of the equations and interpreted as an additional load to be determined by means of iterations. In the second method, the entire loading process is divided into a series of stages at each of which the increments in the resolvent functions satisfy a linear system of equations whose coefficients are the values of the very functions accumulated over all the preceding stages of the loading. However, the applicability of the simple iteration method is limited by its poor rate of convergence, which breaks down at deflections on the order of the thickness, while the method of successive loadings is limited by the error of linearization created in the loading process.

Below we present a method of linearizing a system of resolvent equations of applied shell theory that is free of these drawbacks.

Let us consider a shell of constant thickness h made of a homogeneous isotropic material with modulus of elasticity E and Poisson coefficient ν affected by an arbitrary system of forces with components X_1, X_2 and X_3 in a curvilinear coordinate system $\alpha_1\alpha_2\alpha_3$ (α_1 and α_2 are the directions of the principal curvatures, and α_3 is the normal to the median surface). Within the framework of the applied theory, the system of resolvent equations in a mixed form that describes the elastic equilibrium of the shell may be reduced to the form /5/

$$\begin{gathered}
D\Delta\Delta w + \Delta_k \Phi + L(w, \Phi) = p_1 \qquad (1)\\
H\Delta\Delta\Phi - \Delta_k w - {}^1/_2\, L(w, w) = p_2\\
\Delta w = R_{11}(w) + R_{22}(w);\ \Delta_k w = k_2 R_{11}(w) + k_1 R_{22}(w)\\
L(w, \Phi) = R_{11}(w)\, R_{22}(\Phi) + R_{22}(w)\, R_{11}(\Phi) - 2R_{12}(w)\, R_{12}(\Phi)\\
R_{11}(w) = \frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial\omega}{\partial\alpha_1}\right) + \frac{1}{A_1A_2^2}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_2}\\
R_{22}(w) = \frac{1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial w}{\partial\alpha_2}\right) + \frac{1}{A_1^2A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial w}{\partial\alpha_1}\\
R_{12}(u) = \frac{1}{A_1A_2}\left(\frac{\partial^2 w}{\partial\alpha_1\partial\alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial\omega}{\partial\alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial\omega}{\partial\alpha_2}\right)\\
p_1 = X_3 + (k_1 + k_2)\, U, \quad p_2 = (1 - \nu)\, H\Delta U
\end{gathered}$$

Here w and Φ are the deflection and stress function; D and H, the flexural rigidity and tensile flexibility; A_1 and A_2, Lamé coefficients; U, potential of the tangential load components X_1 and X_2; also k_1 and k_2, principal curvatures.

---

*Prikl.Matem.Mekhan.,Vol.47,No.1,pp.101-107,1983

The system (1) is nonlinear because of the presence of the operators L. Let us represent the nonlinear operator in the first equation of (1) in the form (S is some constant)

$$L(w, \Phi) = SL(\Phi, w) + (1 - S) L(w, \Phi)$$

We introduce the notation

$$\varphi_1 = {}^1/_2 w, \ \varphi_2 = (1 - S) w, \ \varphi_3 = S\Phi \tag{2}$$

The system (1) may be written in the form

$$\begin{aligned} &D\Delta\Delta w + L(\varphi_3, w) + \Delta_k\Phi + L(\varphi_2, \Phi) = p_1 \\ &H\Delta\Delta\Phi - \Delta_k w - L(\varphi_1, w) = p_2 \end{aligned} \tag{3}$$

The quantities φ_i in (3) will be determined through successive approximations; here we take some initial approximation $w^{(0)}, \Phi^{(0)}$ and then determine a value of $\varphi_i^{(0)}$ corresponding to this approximation. Solving the system (3) for given values of $\varphi_i^{(0)}$, we find the next approximation $w^{(1)}, \Phi^{(1)}$, and so on. The process continues until the difference between a succeeding and preceding approximation corresponds to an assigned solution accuracy. Thus at each step of the iteration a linear system of the form (3) with given parameters φ_i must be solved.

We rewrite (3) in the form

$$\begin{aligned} &L_1[w, \Phi] \equiv D\Delta\Delta w + L(\varphi_3, w) + \Delta_2\Phi = p_1 \\ &L_2[w, \Phi] \equiv H\Delta\Delta\Phi - \Delta_1 w = p_2 \\ &\Delta_i w = [k_2 + R_{22}(\varphi_i)] R_{11}(w) + [k_1 + R_{11}(\varphi_i)] R_{22}(w) - \\ &\quad 2R_{12}(\varphi_i) R_{12}(w); \quad i = 1, 2 \end{aligned} \tag{4}$$

The quantity φ_3 in the system (4) is an additional flexural rigidity of the shell, while the additional curvature $R_{jj}(\varphi_i)$ and torsion $R_{12}(\varphi_i)$ parameters for the median surface occur in the operators Δ_i. Consequently, the initial nonlinear system (1) is replaced by a sequence of linear systems of the form (4) with additional rigidity, curvature, and torsion parameters that may be determined by means of iterations. The coefficient S describes the contribution made by the additional flexural rigidity to the overall level of nonlinearity of the system (1). When $S = 0$ in (4), the only additional parameters are the curvature and torsion of the median surface of the shell, which depend on w, i.e., the flexural rigidity does not participate in the iterational process. If $S = 0.5$, the additional curvature and torsion parameters in the two equations of the system (4) are the same ($\Delta_1 = \Delta_2$), and the additional flexural rigidity is determined by the relation $\varphi_3 = 0.5\ \Phi$.

To construct the deformation curve, the algorithm must allow for the replacement of the leading parameters, i.e., it is not the external load which is given, but rather some other parameter of the stress-deformation state of the shell. We represent the external load in the form

$$p_i = Qq_i(\alpha_1, \alpha_2); \ i = 1, 2 \tag{5}$$

where q_i are given functions and Q is an unknown constant.

Since the system (4) is linear, by the principle of superposition its solution may be represented in the form

$$w = Qw_0; \quad \Phi = Q\Phi_0; \quad L_i[w_0, \Phi_0] = q_i; \quad i = 1, 2 \tag{6}$$

From relations (6), it is possible to find the unknown constant Q, which corresponds to a given value of the any leading parameter. Suppose the solution of the linear system of differential equations (4) is determined by the $N + 1$ independent variables z_j, where N of the variables depend on the method of discretization of the problem (in the Bubnov–Galerkin, Rayleigh–Ritz, and collocation methods, these variables will be the coefficients of the corresponding functional series; in the method of finite differences, the values of the resolvent functions at the nodes of the network, etc.), while the $(N + 1)$-th independent variable is the load parameter ($z_{N+1} = Q$). In this case, the system of differential equations (4) is replaced by an algebraic analog is a system of linear algebraic equations

$$\sum_{j=1}^{N} a_{ij}z_j + Qb_i = 0; \quad i = 1, 2, \ldots, N \tag{7}$$

The relations in (6) assume the form

$$\begin{aligned} &z_j = Q\xi_j; \quad j = 1, 2, \ldots, N \\ &\sum_{j=1}^{N} a_{ij}\xi_j + b_i = 0; \quad i = 1, 2, \ldots, N \end{aligned} \tag{8}$$

In constructing the deformation curve in an $(N+1)$-dimensional space Z of variables z_j, the entire loading process is divided into a series of stages at each of which one of the variables z_j or some combination of the variables may be taken as the leading parameter /6/.

Let us suppose that the solution has been found for the two loading stages z_{1j} and z_{2j}. With these solutions we associate two points in the space Z through which we draw a line whose parameter λ is made the leading parameter at the next loading stage. The equation of this line has the form (in parametric expression)

$$z_j = z_{1j} + \lambda\,(z_{2j} - z_{1j}); \quad j = 1, 2, \ldots, N+1 \tag{9}$$

We fix λ and through the point corresponding to this parameter draw the hyperplane perpendicular to the line (9). The equation of this plane has the form

$$\sum_{j=1}^{N+1} (z_{2j} - z_{1j})\,[z_j - z_{1j} - \lambda\,(z_{2j} - z_{1j})] = 0 \tag{10}$$

The solution of the system (7) will be found in the hyperplane (10). Substituting (8) in (10) yields an equation for determining the load parameter:

$$Q = \sum_{j=1}^{N+1} (z_{2j} - z_{1j})\,[z_{1j} + \lambda\,(z_{2j} - z_{1j})]\left[\sum_{j=1}^{N+1} (z_{2j} - z_{1j})\,\xi_j\right]^{-1}, \quad \xi_{N+1} = 1 \tag{11}$$

The relations of (11) and (8) determine the solution of the system (7) at each step of the iteration. By this choice of the leading parameter, we are able to extend the algorithm to all the limiting points and cusps on the deformation curve. The choice of the initial approximation exerts a major influence on the rate of convergence of the iterational process. Following /2/, this approximation may be constructed by extrapolating the solutions from the preceding stages of the loading. For this purpose, we construct an interpolational polynomial and select as our initial approximation the point at which the curve corresponding to this polynomial intersects the hyperplane (10). Clearly, the higher the degree of the approximating polynomial, the better the correspondence between the curve and the true deformation curve and the faster the rate of convergence of the iterational process. In the simplest case, this polynomial may be linear in form, i.e., we select as the initial approximation a point lying on the line (9) and the corresponding value of the leading parameter λ.

At the first stage of the loading, the solution of the linear problem (the system (3) for $\varphi_i = 0$) and the trivial solution $z_{2j} = 0$ may be selected as the z_{1j} and z_{2j}. The numerical value of the parameter λ may be selected as a function of the rate of convergence of the iterational process at the preceding stage of loading. The value $\lambda = 0$ corresponds to the point z_{1j} on the line (9), and the value $\lambda = 1$ to the point z_{2j}. Consequently, at the following loading stage we must have $\lambda > 1$. When $\lambda = 2$, the solution will be found on a hyperplane lying at the same distance d from the point z_{2j} as the point z_{1j} $(d = |z_{2j} - z_{1j}|)$. We fix some number of iterations r_0. If the number of iterations that are required to obtain a solution with an assigned accuracy ε at the preceding loading stage amounts to $r_1 < r_0$, at the next stage we must have $\lambda > 2$, i.e., a solution is found in the hyperplane at a distance $d > |z_{2j} - z_{1j}|$ from the point z_{2j}. Otherwise, $1 < \lambda < 2$ $(d < |z_{2j} - z_{1j}|)$. The relation λ from r_0 and r_1 may be taken, for example in the form

$$\lambda = 2 + \frac{r_0 - r_1}{r_0 + r_1}$$

Here the parameter λ varies over the range $1 < \lambda < 3$, i.e., the maximal distance d from the point z_{2j} to the hyperplane (10) is at most $2\,|z_{2j} - z_{1j}|$. At the first loading stage, the distance between the initial points (the linear and trivial solutions) must be selected such that the influence of geometric nonlinearity would be obviously negligible (for example, with a maximal flexure on the order of $0.2h$).

The convergence of the algorithm at the r-th step of the iteration may be controlled by means of the quantity

$$\sum_{j=1}^{N+1} \left| \frac{z_j^{(r)} - z_j^{(r-1)}}{z_j^{(r-1)}} \right| < \varepsilon$$

To study the convergence of our algorithm, we reviewed the problem of determining the equilibrium of a shell which is hollow above some plane; here it may be assumed that the metric of the median surface coincides with the metric of the plane

$$\alpha_1 = x, \quad \alpha_2 = y, \quad A_1 = A_2 = 1$$

As an example, we consider a proposed rectangle in plane $a \times b$ spherical panel whose edges are hinged and which is affected by a uniformly distributed transverse load p_0 $(p_1 = p_0;$ $p_2 = 0)$. The solution of the linear system (4) is constructed in the double trigonometric series

$$w = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} w_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \tag{12}$$

$$\Phi = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \Phi_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

Substituting (12) in (4) and applying the Bubnov - Galerkin procedure, we arrive at a linear system of algebraic equations of the form (7). Since the stress function in the second equation of the system (4) occurs in the differential operators with constant coefficients, the quantities Φ_{nm} may be eliminated from (7) without any difficulty. Consequently, the only independent variables z_j are only the coefficients w_{nm} of the expansion of the deflection in a Fourier series. The computations are conducted for

$$b = a;\ k_1 = k_2 = 18h/a^2;\ \nu = 0.3$$

We have retained nine terms in the series of (12) $(n, m = 1, 3, 5)$. A further increase in n and m will not lead to a marked change in the results. The initial distance between the points z_{1j} and z_{2j} are selected in such a way that the deflection at the center of the panel amounts of $0.2h$, and the fixed number of iterations is taken as $r_0 = 5$. The influence of the coefficient S in (2) on the rate of convergence of the iterational process was studied.

Table 1

$\frac{w}{h}$	$\frac{p_0 a^4}{Eh^4}$	$S=0$	0.2	0.4	0.6	0.8	1
0.4	88	6	5	5	5	4	4
1.2	151	7	6	5	4	5	6
2.0	127	4	5	5	6	7	12
2.8	98	4	4	4	5	10	64
3.6	76	3	3	4	6	11	—
4.4	67	3	3	4	5	12	—
5.2	91	3	4	4	5	9	—
6.0	140	4	5	7	7	22	—

The results of the computations are presented in the accompanying table; in the first column may be found the values of the deflection at the center of the panel, in the second column the corresponding values of the loading parameter, and in the other columns the number of iterations required to construct the solution to within $\varepsilon = 1\%$ for different values of the coefficient S. From the table, it is clear that on the initial segment of the deformation curve the rate of convergence for all S is virtually the same, but somewhat better when $S = 1$. However, it subsequently speeds up when $S = 0$, and for a deflection $w > 3h$ at $S = 1$, the convergence of the process breaks down.

Thus the optimal value of S (from the standpoint of the rate of convergence of the iterational process) may differ at different segments of the deformation curve and will depend on the particular features of the problem under consideration. It is difficult to predict the optimal value of this parameter, though there are certain general recommendations for selecting it. From the physical standpoint, the quantity $S \cdot 100\%$ indicates the contribution of the supplementary flexural rigidity to the overall level of nonlinearity (or $(1 - S) \cdot 100\%$ which is the proportion of the supplementary curvature and torsion parameters).

This example shows that at low deflections the supplementary flexural rigidity makes the major contribution, that in a neighborhood of the first limiting point the proportions of the additional rigidity and curvature are roughly the same, and that subsequently the additional curvature and torsion parameters play the basic role. In the general case, it is best to select $S = 1$ on the precritical branch; $S = 0.5$ in a nighborhood of the first limiting point; and $S = 0$ in the post-critical stage of deformation. This choice of S may not be optimal, though it must, in all likelihood, ensure a sufficiently rapid rate of convergence of the iterational process.

The initial system of differential equations (4) is linear, therefore our algorithm may be extended to the case of multi-parameter loading. Suppose that a system of M forces is applied to the shell, and that these forces may be related to the load terms p_{ik} in the right side of the equations (1). We represent these quantities in the form

$$p_{ik} = Q_k q_{ik}(\alpha_1, \alpha_2);\quad i = 1, 2;\ k = 1, 2, \ldots, M \tag{13}$$

where q_{ik} are given functions and Q_k are unknown constants.

Since the system (4) is linear, its solution may be represented in the form

$$w = \sum_{k=1}^{M} Q_k w_k; \quad \Phi = \sum_{k=1}^{M} Q_k \Phi_k \qquad (14)$$
$$L_i [w_k, \Phi_k] = q_{ik}; \; i = 1, 2; \; k = 1, 2, \ldots, M$$

In the M-dimensional space Q_k of variables, the curve that describes the loading program may be selected arbitrarily. Suppose the equation of this curve is given in the parametric form

$$Q_k = F_k(t); \quad k = 1, 2, \ldots, M \qquad (15)$$

where t is the parameter of the curve (for example, arc length).

The deformation curve will be constructed in the $(N+1)$-dimensional space Z of variables z_j $(j = 1, 2, \ldots, N)$ and the parameter t $(z_{N+1} = t)$ of the curve (15). In this case the algebraic analog of the system (4) has the form

$$\sum_{j=1}^{N} a_{ij} z_j + \sum_{k=1}^{M} F_k(t)\, b_{ik} = 0; \; i = 1, 2, \ldots, N$$

The relation (14) may be written, using the variables z_j, in the form

$$z_j' = \sum_{k=1}^{M} F_k(t)\, \xi_{jk}; \quad j = 1, 2, \ldots, N \qquad (16)$$
$$\sum_{j=1}^{N} a_{ij} \xi_{jk} + b_{ik} = 0; \quad i = 1, 2, \ldots, N; \quad k = 1, 2, \ldots, M$$

Substituting (16) in the equation of the hyperplane (10) in which the solution will be found yields

$$\sum_{j=1}^{N+1} (z_{2j} - z_{1j}) \Big[\sum_{k=1}^{M} F_k(t)\, \xi_{jk} - z_{1j} - \lambda (z_{2j} - z_{1j}) \Big] = 0 \qquad (17)$$
$$\xi_{N+1,\,k} = 1; \quad k = 1, 2, \ldots, M$$

The latter equation is a nonlinear algebraic equation for the unknown loading parameter t whose solution may be obtained by direct search. For the initial value t_0, we have

$$t_0 = t_1 + \lambda (t_2 - t_1) \qquad (18)$$

which corresponds to the point lying on the line (9) for a given leading parameter $1 < \lambda < 3$. The root t nearest t_0 is found by constructing a procedure for direct search for the equation (17) with initial value (18) in two opposite directions. Substituting the resulting value of t in (16) yields the remaining components of the solution z_j. Thus in the case of multiparametric loading of a shell according to an arbitrarily selected program (15), the entire algorithm remains unchanged, except for the loading parameter, which is determined not from (11) (as in the case of single-parameter loading), but numerically from a nonlinear equation of the form (17).

By means of our algorithm, it is possible to construct a deformation curve in an $(N+1)$-dimensional space Z of variables z_j and the loading parameter t. The initial nonlinear equation (1) must be subjected to a bifurcation in order to localize the general-form branching points (and not just the limiting points) on each loading stage. The linearized system of equations for the perturbed state may be written in the form

$$D\Delta\Delta w + L(\Phi^*, w) + \Delta_k \Phi + L(w^*, \Phi) = 0 \qquad (19)$$
$$H\Delta\Delta\Phi - \Delta_k w - L(w^*, w) = 0$$

where w^*, Φ^* is the solution of the initial nonlinear system (1).

The coefficients of the system (19) coincide, to within a constant multiplier, with the coefficients (2) of the linearized system (3). Consequently, there is no particular difficulty presented by the construction of the algebraic analog of the system (19), since the elements of its matrix a_{ij}^* coincide, to within a constant multiplier, with the elements a_{ij} of the matrix of the system (7). When the determinant $\det(a_{ij}^*)$ vanishes, this will mean that there is a nontrivial solution of the system (19). Thus in constructing the deformation curve it is necessary to determine the sign of the determinant $\det(a_{ij}^*)$ on each loading stage, as a change in this sign indicates the presence of a bifurcation point on the deformation curve. To construct the solution on the secondary branch, it is necessary to introduce a perturbation in the initial approximation based on the obviously trivial components z_j, i.e., to derive a solution from the initial hyperplane of nonzero components.

REFERENCES

1. VLASOV V.Z., General Shell Theory and its Applications to Engineering, Moscow-Leningrad, GOSTEKHIZDAT, 1949.
2. KORNISHIN M.S., Nonlinear Problems of the Theory of Plates and Shallow Shells and Methods of Solution, Moscow, NAUKA, 1964.
3. KRYS'KO V.A., Nonlinear Statics and Dynamics of Nonhomogeneous Shells, Saratov, Izd-vo Sarat. Un-ta, 1976.
4. PETROV V.V., Method of Successsive Loadings in the Nonlinear Theory of Plates and Shallow Shells, Saratov, Izd-vo Sarat. Un-ta, 1975.
5. MUSHTARI Kh.M. and GALIMOV K.Z., Nonlinear Theory of Elastic Shells, Kazan', Tatknigoizdat, 1957.
6. SHALASHILIN V.I., "Certain algorithms for the parametric continuation method in nonlinear problems of the theory of elasticity," In: Nonlinear Plate and Shell Theory, Theses report, Kazan', Izd-vo Kazansk. Un-ta, 1980.

Translated by R.H.S.